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Solutions of Jimbo-Miwa Equation and Konopelchenko-Dubrovsky Equations

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Abstract

The Jimbo-Miwa equation is the second equation in the well known KP hierarchy of integrable systems, which is used to describe certain interesting (3+1)-dimensional waves in physics but not pass any of the conventional integrability tests. The Konopelchenko-Dubrovsky equations arose in physics in connection with the nonlinear weaves with a weak dispersion. In this paper, we obtain two families of explicit exact solutions with multiple parameter functions for these equations by using Xu's stable-range method and our logarithmic generalization of the stable-range method. These parameter functions make our solutions more applicable to related practical models and boundary value problems.

Keywords: Jimbo-Miwa; Konopelchenko-Dubrovsky; Stable-range; Logarithmic stable-range.

AMS Subject Classification (2000): 35Q51, 35C10, 35C15.

1 Introduction

Jimbo and Miwa [6] (1983) first studied the following nonlinear partial differential equation:

$$W_{xxxy} + 3W_{xy}W_x + 3W_yW_{xx} + 2W_{yt} - 3W_{xz} = 0, \tag{1.1}$$

as the second equation in the well known KP hierarchy of integrable systems. The equation is used to describe certain interesting (3+1)-dimensional waves in physics but not pass any of the conventional integrability tests [2]. One of the important features is that the equation has soliton solutions. The space of τ-functions for this hierarchy, given by Jimbo and Miwa [6](1983), is the orbit of the vacuum vector for the Fock representation of the Lie algebra $gl(\infty)$. Dorizzi, Grammaticos, Ramani and Winternitz [2] (1986) calculated Lie symmetries of (1.1) in terms of Lie algebra. They showed that the algebra is infinite dimensional, but does not have the Kac-Moody-Virasoro algebra structure. Rubin

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and Winternitz [11] (1990) found that the joint symmetry algebra of the system of the first two equations in KP hierarchy have a Kac-Moody-Virasoro algebra structure. The generalized W_∞ symmetry algebra of these two equations were found by Lou and Weng [9] (1995). Hong and Oh [4] (2000) got a class of solitary wave solutions of (1.1) by generalizing the tanh method. Fan [3] (2003) obtained a line solitary wave solution, a Jocobi doubly periodic solution and a Weierstrass periodic solution by using modified tanh method. Abdou [1] (2008) found some generalized solitary wave solutions and periodic solutions by the exp-function method.

The equations

$$u_t - u_{xxx} - 6buu_x + \frac{3}{2}a^2u^2u_x - 3v_y + 3au_xv = 0 \tag{1.2a}$$

$$u_y = v_x \tag{1.2b}$$

were introduced by Konopelchenko and Dubrovsky [7] (1984) in connection with the nonlinear weaves with a weak dispersion, where a and b are real constants. These equations can be represented as the commutativity $[L, T] = 0$ of certain differential operators L and T [7]. The system is the two dimensional generalization of the well-known Gardner equation, KP equation (the first equation of the KP hierarchy) and the modified KP equation. Maccari [10] (1999) derived an integrable Davey-Stewartson-type equation from (1.2a) and (1.2b). H. Zhi [22] (2008) found the symmetry group of this system. To solve the Konopelchenko-Dubrovsky equations, various methods have been proposed, such as the standard truncated Painlevé analysis [8], the tanh method and its generalizations [19][21][22], the generalized F-expansion method [15][23][20], the extended Riccati equation rational expansion method [12], exp-function method [1], the tanh-sech method, the cosh-sinh method, the exponential functions method [14] and the homotopy perturbation method [13].

Most of the above existing exact explicit solutions of the Jimbo-Miwa equation and the Konopelchenko-Dubrovsky equations are traveling-wave-type solutions and their slightly generalizations. These solutions do not fully reflect the features of these nonlinear partial differential equations. It is desirable to find new exact explicit solutions that capture more features of these equations.

Using certain finite-dimensional stable range of the nonlinear term, Xu [16] found a family of exact solutions with seven parameter functions for the equation of nonstationary transonic gas flows, which blow up on a moving line. Moreover, he [17] solved the short wave equation and the Khokhlov-Zabolotskaya equations by the same method and obtained certain interesting singular and smooth explicit exact solutions with multiple parameter functions.

In this paper, we find two families of explicit exact solutions with multiple parameter functions for the Jimbo-Miwa equation and the Konopelchenko-Dubrovsky equations by using Xu's stable-range method and our logarithmic generalization of the stable-range method, motivated from the standard truncated Painlevé analysis, as in [8]. The first family of solutions are polynomial in the variable x or y. The second family solutions are not polynomial in any variable. They are logarithms of the functions that are polynomial either in x or in y.

Our solutions in general are not traveling-wave-type solutions. Their multiple-parameter-function feature makes them more applicable to related practical models and boundary value problems.

In Section 2, we find exact solutions of the the Jimbo-Miwa equation. The Konopelchenko-Dubrovsky equations are solved in Section 3.

2 Solutions of the Jimo-Miwa Equation

2.1 Stable-Range Approach

We assume that

$$W = \sum_{m=0}^{n} A_m(y, z, t)x^m \tag{2.1}$$

is a solution of the Jimbo-Miwa equation (1.1). First we consider the case $n \leq 2$, i.e.

$$W = A_2 x^2 + A_1 x + A_0. \tag{2.2}$$

Note that

$$W_x = 2A_2 x + A_1, \quad W_{xx} = 2A_2, \quad W_{xxx} = 0, \tag{2.3}$$

$$W_y = A_{2y}x^2 + A_{1y}x + A_{0y}, \quad W_{xy} = 2A_{2y}x + A_{1y}, \tag{2.4}$$

and

$$W_{xz} = 2A_{2z}x + A_{1z}, \quad W_{yt} = A_{2yt}x^2 + A_{1yt}x + A_{0yt}. \tag{2.5}$$

Substituting (2.2)-(2.5) into (1.1), we get

$$\begin{aligned} & 3(2A_{2y}x + A_{1y})(2A_2 x + A_1) + 3(A_{2y}x^2 + A_{1y}x + A_{0y}) \cdot 2A_2 \\ + \quad & 2(A_{2yt}x^2 + A_{1yt}x + A_{0yt}) - 3(2A_{2z}x + A_{1z}) = 0. \end{aligned} \tag{2.6}$$

Thus

$$9A_2 A_{2y} + A_{2yt} = 0, \tag{2.7a}$$

$$3A_1 A_{2y} + 6A_{1y}A_2 + A_{1yt} - 3A_{2z} = 0, \tag{2.7b}$$

$$3A_1 A_{1y} + 6A_2 A_{0y} + 2A_{0yt} - 3A_{1z} = 0. \tag{2.7c}$$

Observe that

$$A_2 = \alpha_t(t, z) \tag{2.8}$$

and

$$A_2 = (\frac{9}{2}t + \beta(y, z))^{-1} \tag{2.9}$$

are solutions of (2.7a), where α and β are arbitrary differential functions. Throughout this paper, the indefinite integration means an antiderivative of the integrand with respect to the integral variable. Substituting (2.8) into (2.7b), we get

$$6\alpha_t A_{1y} + A_{1yt} - 3\alpha_{tz} = 0. \tag{2.10}$$

It implies

$$A_1 = e^{-6\alpha}\left(\gamma(y,z) + 3y\int \alpha_{tz}e^{6\alpha}\mathrm{d}t\right) + \rho(z,t), \tag{2.11}$$

where $\gamma(y,z)$ and $\rho(z,t)$ are arbitrary functions. Similarly, we get

$$A_0 = e^{-3\alpha}\left(\eta(y,z) + \frac{3}{2}\iint (A_{1z} - A_1A_{1y})e^{3\alpha}\mathrm{d}t\mathrm{d}y\right) + \zeta(z,t), \tag{2.12}$$

where $\eta(y,z)$ and $\zeta(z,t)$ are arbitrary functions.

Theorem 2.1. *For arbitrary functions $\alpha(t,z)$, $\eta(y,z)$, $\gamma(y,z)$, $\rho(z,t)$ and $\zeta(z,t)$, we have the solution*

$$\begin{aligned} W &= \alpha_t(t,z)x^2 + A_1x + e^{-3\alpha}(\eta(y,z) \\ &\quad + \frac{3}{2}\int\int (A_{1z} - A_1A_{1y})e^{3\alpha}\mathrm{d}t\mathrm{d}y) + \zeta(z,t) \end{aligned} \tag{2.13}$$

of the Jimbo-Miwa equation (1.1), *where A_1 is given in* (2.11).

Next we deal with $A_2 = (\frac{9}{2}t + \beta(y,z))^{-1}$. Assume

$$A_1 = \sum_{n\in\mathbb{Z}} B_n(y,z)(\frac{9}{2}t + \beta(y,z))^n. \tag{2.14}$$

Then

$$A_{1y} = \sum_{n\in\mathbb{Z}} (B_{ny} + (n+1)B_{n+1}\beta_y)(\frac{9}{2}t + \beta(y,z))^n \tag{2.15}$$

and

$$A_{1yt} = \sum_{n\in\mathbb{Z}} \frac{9}{2}(n+1)(B_{(n+1)y} + (n+2)B_{n+2}\beta_y)(\frac{9}{2}t + \beta(y,z))^n. \tag{2.16}$$

Thus by (2.7b), we have

$$\begin{aligned} &\sum_{n\in\mathbb{Z}}(\frac{(n+3)(3n+4)}{2}B_{n+2}\beta_y + \frac{3n+7}{2}B_{(n+1)y})(\frac{9}{2}t + \beta(y,z))^n \\ = \ &\frac{-\beta_z}{(\frac{9}{2}t + \beta(y,z))^2}. \end{aligned} \tag{2.17}$$

Hence

$$-B_0\beta_y + \frac{1}{2}B_{(-1)y} = -\beta_z, \qquad n = -2 \tag{2.18a}$$

$$\frac{(n+3)(3n+4)}{2}B_{n+2}\beta_y + \frac{3n+7}{2}B_{(n+1)y} = 0. \qquad n \neq -2 \tag{2.18b}$$

Let $n=-3$ in (2.18b). We have $B_{(-2)y}=0$. Then

$$B_{-2}=\gamma_{-2}(z), \tag{2.19}$$

where γ_{-2} is an arbitrary function. Thus, by (2.19),

$$B_{-l-2}=\sum_{m=0}^{l}\frac{3l+5}{3m+5}\binom{l}{m}\gamma_{-2-m}(z)\beta^{l-m}, \tag{2.20}$$

where $\gamma_{-2-m}(z)$ are arbitrary functions.

If $\beta_y=0$, then

$$B_{l-1}=\gamma_{l-1}(z), \quad \text{for } l\geq 1, \tag{2.21}$$

and

$$B_{-1}=-2\beta_z y+\gamma_{-1}(z). \tag{2.22}$$

It is not interesting. Thus, we assume that $\beta_y\neq 0$. Let $B_n=\gamma_n(z)$ for $n\in\mathbb{N}$. Then $B_{(n+1)}=0$, and we have that

$$B_m=\sum_{r=0}^{n-m}(-1)^{n-m+r}\frac{3m+1}{3(n-r)+1}\binom{n+1-r}{m+1}\beta^{n-m-r}\gamma_{n-r}(z) \tag{2.23}$$

for $0\leq m\leq n$.

$$\begin{aligned} B_{-1} &= 2\int B_0\beta_y\mathrm{d}y-2\int\beta_z\mathrm{d}y+\gamma_{-1}(z)\\ &= 2\sum_{r=0}^{n}(-1)^{n+r}\frac{1}{3(n-r)+1}\gamma_{n-r}(z)\beta^{n-r+1}-2\int\beta_z\mathrm{d}y+\gamma_{-1}(z). \end{aligned} \tag{2.24}$$

Hence, we get that

$$\begin{aligned} A_0 &= \int(\frac{9}{2}t+\beta)^{-\frac{2}{3}}\eta(y,z)\mathrm{d}y\\ &\quad +\int(\frac{9}{2}t+\beta)^{-\frac{2}{3}}(\int(\frac{9}{2}t+\beta)^{\frac{2}{3}}(A_{2z}-A_1A_1y)\mathrm{d}t)\mathrm{d}y+\zeta(z,t). \end{aligned} \tag{2.25}$$

Theorem 2.2. *For arbitrary functions $\beta(y,z)$, $\gamma_s(z)$, $\eta(y,z)$ and $\zeta(z,t)$, the function*

$$\begin{aligned} W &= (\frac{9}{2}t+\beta(y,z))^{-1}x^2+A_1x+\int(\frac{9}{2}t+\beta)^{-\frac{2}{3}}\eta(y,z)\mathrm{d}y\\ &\quad +\int(\frac{9}{2}t+\beta)^{-\frac{2}{3}}(\int(\frac{9}{2}t+\beta)^{\frac{2}{3}}(-\beta_z(\frac{9}{2}t+\beta)^{-2}-A_1A_{1y})\mathrm{d}t)\mathrm{d}y\\ &\quad +\zeta(z,t) \end{aligned} \tag{2.26}$$

is a solution of the Jimbo-Miwa equation (1.1), *where*

$$\begin{aligned} A_1 &= (\sum_{r=0}^{n}(-1)^{n+r}\frac{2\gamma_{n-r}(z)}{3(n-r)+1}\beta^{n-r+1} - 2\int\beta_z \mathrm{d}y + \gamma_{-1}(z))(\frac{9}{2}t+\beta)^{-1} \\ &+ \sum_{m=0}^{n}\sum_{r=0}^{n-m}(-1)^{n-m+r}\frac{3m+1}{3(n-r)+1}\binom{n+1-r}{m+1} \\ &\times\beta^{n-m-r}\gamma_{n-r}(z)(\frac{9}{2}t+\beta)^m. \end{aligned} \tag{2.27}$$

Now we consider the case $n \geq 3$ in (2.1). In this case,

$$A_m = A_m(z,t) \qquad \text{for} \quad m = 2,\ldots,n. \tag{2.28}$$

We have

$$\begin{aligned} & 3\sum_{m=0}^{n-1}(m+1)^2 A_{1y}A_{m+1}x^m + 3\sum_{m=0}^{n-2}(m+2)(m+1)A_{0y}A_{m+2}x^m \\ - & 3\sum_{m=0}^{n-1}(m+1)A_{(m+1)z}x^m + 2A_{1yt}x + 2A_{0yt} = 0, \end{aligned} \tag{2.29}$$

by (1.1). Note that

$$A_{1y} = \frac{A_{nz}}{nA_n}, \tag{2.30}$$

which implies

$$A_1 = \frac{A_{nz}}{nA_n}y + \eta(z,t), \tag{2.31}$$

and

$$A_{(m+1)z} = (m+1)A_{1y}A_{m+1} + (m+2)A_{0y}A_{m+2}, \quad \text{for } m = 2,\ldots,n, \tag{2.32a}$$

$$A_{2z} = 2A_{1y}A_2 + 3A_{0y}A_3 + \frac{1}{3}A_{1yt}, \tag{2.32b}$$

$$A_{1z} = A_{1y}A_1 + 2A_{0y}A_2 + \frac{2}{3}A_{0yt}, \tag{2.32c}$$

where $A_{n+r} = 0$ for $r > 0$. Then

$$(\frac{A_{nz}}{nA_n})^2 - (\frac{A_{nz}}{nA_n})_z = 0. \tag{2.33}$$

So, we get that

$$A_n = \gamma_n(t)(-z+g(t))^{-n}, \tag{2.34}$$

where $\gamma_n(t)$ and $g(t)$ are arbitrary functions. By induction, we obtain that

$$A_{n-m} = \frac{\prod_{s=0}^{m-1}(n-s)}{(-z+g)^{n-m}}\sum_{s=0}^{m}\gamma_{n-s}(t)\frac{(\int\frac{\varphi}{-z+g}\mathrm{d}z)^{m-s}}{(m-s)!} \tag{2.35}$$

for $m = 0, \ldots, n-3$, where $\varphi = A_{0y}$, and $\gamma_{n-s}(t)$ are arbitrary functions. By (2.32b),

$$\begin{aligned} A_2 \;=\; & (\prod_{s=0}^{n-3}(n-s))(-z+g)^{-2}\sum_{s=0}^{n-2}\gamma_{n-s}\frac{(\int \frac{\varphi}{-z+g}\mathrm{d}z)^{n-2-s}}{(n-2-s)!} \\ & -\frac{g_t z}{3}(-z+g)^{-2} \end{aligned} \tag{2.36}$$

Moreover, by (2.32c), we have

$$\eta = \frac{1}{-z+g}(2\int(\varphi A_2 + \frac{\varphi_t}{3})(-z+g)\mathrm{d}z + h(t)), \tag{2.37}$$

where $h(t)$ ia an arbitrary function. Then by (2.31), we can get the explicit form of A_1. Moreover, $A_{0y} = \varphi$. Integrate the function $\varphi(z,t)$, we obtain that

$$A_0 = y\varphi(z,t) + f(z,t), \tag{2.38}$$

where f is an arbitrary function.

Theorem 2.3. *Let The $n > 2$ be an integer, and let $g(t)$, $\gamma_s(t)$, $\varphi(z,t)$, $f(z,t)$ and $h(t)$ be arbitrary functions. Then the function*

$$\begin{aligned} W \;=\; & \sum_{m=0}^{n-2}(\frac{\prod_{s=0}^{m-1}(n-s)}{(-z+g(t))^{n-m}}\sum_{s=0}^{m}\gamma_{n-s}(t)\frac{(\int\frac{\varphi}{-z+g}\mathrm{d}z)^{m-s}}{(m-s)!})x^{n-m} \\ & -\frac{g_t}{3}(-z+g)^{-2}x^2 + \frac{(y+h(t))x}{-z+g} \\ & +\frac{2x}{-z+g}(\int(\frac{\varphi_t}{3} + \varphi((\prod_{s=0}^{n-3}(n-s))(-z+g)^{-2}\sum_{s=0}^{n-2}\gamma_{n-s} \\ & \times\frac{(\int\frac{\varphi}{-z+g}\mathrm{d}z)^{n-2-s}}{(n-2-s)!} - \frac{g_t z\varphi}{3}(-z+g)^{-2}))(-z+g)\mathrm{d}z) \\ & +y\varphi(z,t) + f(z,t) \end{aligned} \tag{2.39}$$

is a solution of the Jimbo-Miwa equation (1.1).

Suppose

$$W = A(x,z,t)y + B(x,z,t). \tag{2.40}$$

Then

$$W_x = A_x y + B_x, \quad W_y = A, \quad W_{xx} = A_{xx}y + B_{xx}, \quad W_{xy} = A_x, \tag{2.41}$$

$$W_{xz} = A_{xz}y + B_{xz}, \; W_{yt} = A_t, \; W_{xxx} = A_{xxx}y + B_{xxx}, \; W_{xxxy} = A_{xxx}. \tag{2.42}$$

Substituting (2.40)-(2.42) into (1.1), we get

$$A_x^2 + AA_{xx} - A_{xz} = 0, \tag{2.43a}$$

$$A_{xxx} + 3A_xB_x + 3AB_{xx} + 2A_t - 3B_{xz} = 0. \tag{2.43b}$$

Note that (2.43a) is the x-derivative inviscid Burgers equation [5]. A solution is

$$A = -\frac{x - c(t)}{z - d(t)}, \tag{2.44}$$

where $c(t)$ and $d(t)$ are arbitrary functions.

Substituting (2.44) into (2.43b), we obtain

Theorem 2.4. *The function*

$$W = -\frac{x - c(t)}{z - d(t)} y + e^{\left(\frac{x - c(t)}{z - d(t)}\right)} \varphi(t) - \frac{1}{3} d'(t) \frac{(x - c(t))^2}{z - d(t)} + \frac{2}{3} c'(t) x + f(t, z) \tag{2.45}$$

is a solution of the Jimbo-Miwa equation (1.1) *for arbitrary functions* $c(t)$, $d(t)$, $\varphi(t)$ *and* $f(t, z)$.

Assume

$$W = A(x, z, t) y^2 + B(x, z, t) y + D(x, z, t). \tag{2.46}$$

Then

$$W_x = A_x y^2 + B_x y + D_x, \quad W_y = 2Ay + B, \tag{2.47}$$

$$W_{xx} = 2A_{xx} y^2 + B_{xx} y + D_{xx}, \quad W_{xy} = 2A_x y + B_x, \tag{2.48}$$

$$W_{xz} = A_{xz} y^2 + B_{xz} y + D_{xz}, \quad W_{yt} = 2A_t y + B_t, \tag{2.49}$$

and

$$W_{xxx} = A_{xxx} y^2 + B_{xxx} y + D_{xxx}, \quad W_{xxxy} = 2A_{xxx} y + B_{xxx}. \tag{2.50}$$

Substituting (2.46)-(2.50) into (1.1), we get

$$A_x^2 + AA_{xx} = 0, \tag{2.51a}$$

$$3A_x B_x + 2AB_{xx} + A_{xx} B - A_{xz} = 0, \tag{2.51b}$$

$$2A_{xxx} + 3(2A_x D_x + B_x^2) + 3(2AD_{xx} + BB_{xx}) + 4A_t - 3B_{xz} = 0, \tag{2.51c}$$

$$B_{xxx} + 3B_x D_x + 3BD_{xx} + 2B_t - 3D_{xz} = 0. \tag{2.51d}$$

Observe that

$$A = (bx + c)^{\frac{1}{2}} \tag{2.52}$$

is a solution of (2.51a) for arbitrary functions $b = b(z, t)$ and $c = c(z, t)$. Suppose

$$B = \sum_{n \in \mathbb{Z}} a_n(z, t)(bx + c)^n. \tag{2.53}$$

By (2.51b), we have

$$B = \frac{b_z}{5b^2}(bx + c) + \frac{bc_z - b_z c}{b^2}. \tag{2.54}$$

Denote $f = D_x$. Then (2.51c) and (2.51d) become

$$2A_{xxx} + 6(Af)_x + 3(BB_x)_x + 4A_t - 3B_{xz} = 0, \tag{2.55a}$$

$$3(Bf)_x + 2B_t - 3f_z = 0. \tag{2.55b}$$

Let

$$\xi = bx + c. \tag{2.56}$$

We have

$$A = \xi^{\frac{1}{2}} \tag{2.57}$$

by (2.52), and

$$\xi_x = b, \quad \xi_z = \frac{b_z}{b}\xi + b(\frac{c}{b})_z, \quad \xi_t = \frac{b_t}{b}\xi + b(\frac{c}{b})_t. \tag{2.58}$$

Note that

$$B = \frac{b_z}{5b^2}\xi + (\frac{c}{b})_z, \quad B_x = \frac{b_z}{5b}, \quad B_{xz} = \frac{1}{5}(\frac{b_z}{b})_z, \tag{2.59}$$

and

$$\begin{aligned} A_t &= \frac{1}{2}\xi^{-\frac{1}{2}}(\frac{b_t}{b}\xi + b(\frac{c}{b})_t) \\ &= \frac{b_t}{2b}\xi^{\frac{1}{2}} + \frac{b}{2}(\frac{c}{b})_t\xi^{-\frac{1}{2}}. \end{aligned} \tag{2.60}$$

Hence, by (2.55a),

$$f = -\frac{1}{6}(\frac{4}{3}\frac{b_t}{b^2}\xi + \frac{1}{b}(-\frac{3}{5}(\frac{b_z}{b})_z + \frac{3}{25}\frac{b_z^2}{b^2})\xi^{\frac{1}{2}} + 4(\frac{c}{b})_t - \frac{b^2}{2}\xi^{-2}) + g\xi^{-\frac{1}{2}}, \tag{2.61}$$

where $g = g(z,t)$. Substituting (2.61) into (2.55b) and checking the coefficients of ξ^{-2}, we get

$$b_z = 0. \tag{2.62}$$

Then

$$f = -\frac{1}{6}(\frac{4}{3}\frac{b_t}{b^2}\xi + 4(\frac{c}{b})_t - \frac{b^2}{2}\xi^{-2} - 6g\xi^{-\frac{1}{2}}), \tag{2.63}$$

and

$$B = \frac{c_z}{b}. \tag{2.64}$$

Comparing the coefficients of the polynomials with respect to ξ in the two sides of (2.55b), we get

$$c_{zt}b = b_t c_z, \tag{2.65a}$$

$$g_z = 0. \tag{2.65b}$$

Moreover,

$$c = h(z)b(t) + \eta(t), \quad g = g(t), \tag{2.66}$$

and

$$D = -\frac{1}{6}(\frac{b_t}{3b^3}\xi^2 + 2(\frac{\eta}{b})_t\frac{\xi}{b} - 3\frac{g}{b}\xi^{\frac{1}{2}} + \frac{b}{2}\xi^{-1}) + l(z,t). \tag{2.67}$$

Theorem 2.5. *For arbitrary functions $b(t)$, $h(z)$, $\eta(t)$, $g(t)$ and $l(z,t)$, the function*

$$\begin{aligned} W &= (b(t)x+h(z)b(t)+\eta(t))^{\frac{1}{2}}y^2+h_z y \\ &\quad -\frac{1}{6}(\frac{b_t}{3b^3}(b(t)x+h(z)b(t)+\eta(t))^2+2(\frac{\eta(t)}{b(t)})_t\frac{b(t)x+h(z)b(t)+\eta(t)}{b(t)} \\ &\quad -3\frac{g(t)}{b(t)}(b(t)x+h(z)b(t)+\eta(t))^{\frac{1}{2}}+\frac{b(t)}{2}(b(t)x+h(z)b(t)+\eta(t))^{-1}) \\ &\quad +l(z,t) \end{aligned} \tag{2.68}$$

is a solution of the Jimbo-Miwa equation (1.1).

Let

$$W = A(x,z,t)y^n + B(x,z,t)y + C(x,z,t), \tag{2.69}$$

where $n \geq 3$. Then

$$W_x = A_x y^n + B_x y + C_x, \quad W_y = nAy^{n-1} + B, \tag{2.70}$$

$$W_{xx} = A_{xx}y^n + B_{xx}y + C_{xx}, \quad W_{xy} = nA_x y^{n-1} + B_x, \tag{2.71}$$

$$W_{xz} = A_{xz}y^n + B_{xz}y + C_{xz}, \quad W_{yt} = nA_t y^{n-1} + B_t, \tag{2.72}$$

and

$$W_{xxxy} = nA_{xxx}y^{n-1} + B_{xxx}. \tag{2.73}$$

Substituting (2.69)-(2.73) into (1.1), we get

$$\begin{aligned} & nA_{xxx}y^{n-1} + B_{xxx} + 3n(A_x^2 + AA_{xx})y^{2n-1} \\ + \;& 3((n+1)A_xB_x + nAB_{xx} + A_{xx}B)y^n + 3n(A_xC_x + AC_{xx})y^{n-1} \\ + \;& 3(B_x^2 + BB_{xx})y + 3(B_xC_x + BC_{xx}) + 2nA_ty^{n-1} + 2B_t \\ - \;& 3A_{xz}y^n - 3B_{xz}y - 3C_{xz} = 0. \end{aligned} \tag{2.74}$$

i.e.

$$A_x^2 + AA_{xx} = 0, \tag{2.75a}$$

$$nAB_{xx} + (n+1)A_xB_x + A_{xx}B - A_{xz} = 0, \tag{2.75b}$$

$$A_{xxx} + 3(AC_x)_x + 2A_t = 0, \tag{2.75c}$$

$$B_x^2 + BB_{xx} - B_{xz} = 0, \tag{2.75d}$$

$$B_{xxx} + 3(BC_x)_x + 2B_t - 3C_{xz} = 0. \tag{2.75e}$$

Hence

$$A = (\phi(z,t)x + \psi(z,t))^{\frac{1}{2}} := \xi^{\frac{1}{2}}. \tag{2.76}$$

Note that

$$A_x = \frac{1}{2}\phi\xi^{-\frac{1}{2}}, \quad A_{xx} = -\frac{\phi^2}{4}\xi^{-\frac{3}{2}}, \quad A_{xxx} = \frac{3}{8}\phi^3\xi^{-\frac{5}{2}}, \tag{2.77}$$

$$A_t = \frac{1}{2}\frac{\phi_t}{\phi}\xi^{\frac{1}{2}} + \frac{1}{2}\frac{\phi\psi_t - \phi_t\psi}{\phi}\xi^{-\frac{1}{2}}, \tag{2.78}$$

and

$$A_{xz} = \frac{1}{4}\phi_z\xi^{-\frac{1}{2}} - \frac{1}{4}(\phi\psi_z - \phi_z\psi)\xi^{-\frac{3}{2}}. \tag{2.79}$$

Set

$$B = \sum_{m\in\mathbb{Z}} a_m\xi^m. \tag{2.80}$$

Then

$$B_x = \sum_{m\in\mathbb{Z}} m\phi a_m\xi^{m-1}, \quad B_{xx} = \sum_{m\in\mathbb{Z}} m(m-1)\phi^2 a_m\xi^{m-2}. \tag{2.81}$$

Thus

$$\sum_{m\in\mathbb{Z}}(nm(m-1) + \frac{(n+1)}{2}m - \frac{1}{4})a_m\phi^2\xi^m = \frac{1}{4}(\phi_z\psi - \phi\psi_z), \tag{2.82}$$

i.e.

$$\frac{2n+1}{4}a_1\phi^2 = \frac{1}{4}\phi_z, \tag{2.83a}$$

$$-\frac{1}{4}a_0\phi^2 = -\frac{1}{4}(\phi\psi_z - \phi_z\psi), \tag{2.83b}$$

and

$$a_m = 0, \quad \text{if } \ m \neq \ 0 \text{ or } 1. \tag{2.84}$$

It deduces to

$$B = \frac{\phi_z}{(2n+1)\phi^2}\xi + (\frac{\psi}{\phi})_z. \tag{2.85}$$

By (2.75c), we get

$$\begin{aligned}(AC_x)_x &= -\frac{1}{3}(A_{xxx} + 2A_t) \\ &= -\frac{1}{3}(\frac{\phi_t}{\phi}\xi^{\frac{1}{2}} + \frac{\phi\psi_t - \phi_t\psi}{\phi}\xi^{-\frac{1}{2}} + \frac{3}{8}\phi^3\xi^{-\frac{5}{2}}).\end{aligned} \tag{2.86}$$

Thus

$$C_x = -\frac{1}{3}(\frac{2}{3}\frac{\phi_t}{\phi}\xi + 2(\frac{\psi}{\phi})_t + f\xi^{-\frac{1}{2}} - \frac{1}{4}\phi^2\xi^{-2}). \tag{2.87}$$

Moreover,

$$\begin{aligned}3BC_x &= -(\frac{\phi_z\phi_t}{(2n+1)\phi^4}\xi^2 + (2(\frac{\psi}{\phi})_t\frac{\phi_z}{(2n+1)\phi^2} + \frac{2}{3}(\frac{\psi}{\phi})_z\frac{\phi_t}{\phi^2})\xi + \frac{f\phi_z}{(2n+1)\phi^2}\xi^{\frac{1}{2}} \\ &\quad +2(\frac{\psi}{\phi})_z\frac{\psi}{\phi})_t + f(\frac{\psi}{\phi})_z\xi^{\frac{1}{2}} - \frac{1}{4(2n+1)}\phi_z\xi^{-1} - \frac{1}{4}\phi^2(\frac{\psi}{\phi})_z\xi^{-2}),\end{aligned} \tag{2.88}$$

$$
\begin{aligned}
3(BC_x)_x &= -(\frac{2\phi_z\phi_t}{(2n+1)\phi^3}\xi + \frac{1}{\phi}(\frac{2}{2n+1}\phi_z(\frac{\psi}{\phi})_t + \frac{2}{3}(\frac{\psi}{\phi})_z\phi_t) \\
&\quad + \frac{1}{2}\frac{f\phi_z}{(2n+1)\phi}\xi^{-\frac{1}{2}} - \frac{1}{2}f(\frac{\psi}{\phi})_z\phi\xi^{-\frac{3}{2}} + \frac{\phi_z\phi}{4(2n+1)}\xi^{-2} \\
&\quad + \frac{1}{2}\phi^3(\frac{\psi}{\phi})_z\xi^{-3}),
\end{aligned} \tag{2.89}
$$

$$
\begin{aligned}
2B_t &= \frac{2}{2n+1}(\frac{\phi_z}{\phi^2})_t\xi + \frac{2}{2n+1}\frac{\phi_z}{\phi^2}\xi_t + 2(\frac{\psi}{\phi})_{zt} \\
&= \frac{2}{2n+1}\frac{\phi_{zt}\phi - \phi_z\phi_t}{\phi^3}\xi + \frac{2}{2n+1}\frac{\phi_z(\phi\psi_t - \phi_t\psi)}{\phi^3} + 2(\frac{\psi}{\phi})_{zt},
\end{aligned} \tag{2.90}
$$

$$
\begin{aligned}
-3C_{xz} &= \frac{2}{3}((\frac{\phi_t}{\phi^2})_z\xi + \frac{\phi_t}{\phi^2}(\frac{\phi_z}{\phi}\xi + \frac{\phi\psi_z - \phi_z\psi}{\phi})) + 2(\frac{\psi}{\phi})_{zt} + f_z\xi^{-\frac{1}{2}} \\
&\quad - \frac{1}{2}f\xi^{-\frac{3}{2}}(\frac{\phi_z}{\phi}\xi + \frac{\phi\psi_z - \phi_z\psi}{\phi}) \\
&\quad + \frac{1}{2}(-\phi\phi_z\xi^{-2} + \phi^2\xi^{-3}(\frac{\phi_z}{\phi}\xi + \frac{\phi\psi_z - \phi_z\psi}{\phi})) \\
&= \frac{2}{3}\frac{\phi_{tz}\phi - \phi_t\phi_z}{\phi^3}\xi + \frac{2}{3}\frac{\phi_t}{\phi^3}(\phi\psi_z - \phi_z\psi) + 2(\frac{\psi}{\phi})_{zt} + (f_z - \frac{1}{2}\frac{f\phi_z}{\phi})\xi^{-\frac{1}{2}} \\
&\quad - \frac{1}{2}\frac{f(\phi\psi_z - \phi_z\psi)}{\phi}\xi^{-\frac{3}{2}} + \frac{1}{2}\phi(\phi\psi_z - \phi_z\psi)\xi^{-3}.
\end{aligned} \tag{2.91}
$$

Hence

$$\phi_z = 0. \tag{2.92}$$

Furthermore,

$$3(BC_x)_x = -\frac{2}{3}\frac{\psi_z\phi_t}{\phi^2} + \frac{1}{2}f\psi_z\xi^{-\frac{3}{2}} - \frac{1}{2}\phi^2\psi_z\xi^{-3}, \tag{2.93}$$

$$2B_t = 2(\frac{\psi_z}{\phi})_t, \tag{2.94}$$

and

$$-3C_{xz} = \frac{2}{3}\frac{\phi_t\psi_z}{\phi^2} + 2(\frac{\psi_z}{\phi})_t + f_z\xi^{-\frac{1}{2}} - \frac{1}{2}\phi^2 f\psi\xi^{-\frac{3}{2}}. \tag{2.95}$$

Thus

$$(\frac{\psi_z}{\phi})_t = 0, \qquad f_z = 0. \tag{2.96}$$

We get

$$\psi = \phi(t)h(z) + g(t), \quad \text{and} \quad f = f(t). \tag{2.97}$$

So

$$A = (\phi x + \phi(t)h(z) + g)^{\frac{1}{2}}, \quad \text{and} \quad B = h_z. \tag{2.98}$$

$$C = -\frac{1}{6}(\frac{\phi_t}{3\phi^3}\xi^2 + 2(\frac{g}{\phi})_t\frac{\xi}{\phi} - 3\frac{f}{\phi}\xi^{\frac{1}{2}} + \frac{\phi}{2}\xi^{-1}) + \eta(z,t). \tag{2.99}$$

Together with Theorem 2.5, we get that

Theorem 2.6. *Let $\phi(t)$, $h(z)$, $g(t)$, $f(t)$ and $\eta(z,t)$ are arbitrary functions, and let $n \geq 2$ be an integer. Then the function*

$$\begin{aligned} W &= (\phi x + \phi h + g)^{\frac{1}{2}} y^n + h_z y - \frac{1}{6}(\frac{1}{3}\frac{\phi_t}{\phi^3}(\phi x + \phi h + g)^2 \\ &\quad + 2(\frac{g}{\phi})_t \frac{\phi x + \phi h + g}{\phi} - 3\frac{f}{\phi}(\phi x + \phi h + g)^{\frac{1}{2}} \\ &\quad + \frac{1}{2}\phi(\phi x + \phi h + g)^{-1}) + \eta(z,t) \end{aligned} \tag{2.100}$$

is a solution of the Jimbo-Miwa equation (1.1).

2.2 Logarithmic Stable-Range Approach

Suppose

$$W = a(\log f)_x = a\frac{f_x}{f}, \tag{2.101}$$

for some constant a and some function f in t, x, y, z. Then

$$W_x = a\frac{f f_{xx} - f_x^2}{f^2}, \quad W_{xx} = a\frac{f^2 f_{xxx} - 3f f_x f_{xx} + 2f_x^3}{f^3}, \tag{2.102}$$

$$W_{xxx} = a\frac{f^3 f_{xxxx} - 4f^2 f_x f_{xxx} - 3f^2 f_{xx}^2 + 12f f_x^2 f_{xx} - 6f_x^4}{f^4}, \tag{2.103}$$

$$\begin{aligned} W_{xxxxy} &= \frac{a}{f^5}(f^4 f_{xxxxy} - f^3(f_{xxxx}f_y + 4f_{xxx}f_{xy} + 6f_{xx}f_{xxy} + 4f_x f_{xxxy}) \\ &\quad + f^2(8f_x f_{xxx} f_y + 6f_{xx}^2 f_y + 24 f_x f_{xx} f_{xy} + 12 f_x^2 f_{xxy}) \\ &\quad - f(36 f_x^2 f_{xx} f_y + 24 f_x^3 f_{xy}) + 24 f_x^4 f_y), \end{aligned} \tag{2.104}$$

$$W_{xy} = \frac{a}{f^3}(f^2 f_{xxy} - f(f_{xx}f_y + 2f_x f_{xy}) + 2f_x^2 f_y), \tag{2.105}$$

$$W_{xz} = \frac{a}{f^3}(f^2 f_{xxz} - f(f_{xx}f_z + 2f_x f_{xz}) + 2f_x^2 f_z), \tag{2.106}$$

$$W_y = \frac{a}{f^2}(f f_{xy} - f_x f_y), \tag{2.107}$$

and

$$W_{yt} = \frac{a}{f^3}(f^2 f_{xyt} - f(f_{xy}f_t + f_x f_{yt} + f_y f_{xt}) + 2f_x f_y f_t). \tag{2.108}$$

Substituting (2.101)-(2.108) into (1.1), we find

$$\begin{aligned}
& f_{xxxxy}f^4 - (f_{xxxx}f_y + 4f_{xxx}f_{xy} + 6f_{xx}f_{xxy} + 4f_x f_{xxxy})f^3 \\
+ \quad & (8f_x f_{xxx}f_y + 6f_{xx}^2 f_y + 24f_x f_{xx}f_{xy} + 12f_x^2 f_{xxy})f^2 \\
- \quad & (36f_x^2 f_{xx}f_y + 24f_x^3 f_{xy})f + 24f_x^4 f_y \\
+ \quad & 3a(f^2 f_{xxy} - f(f_{xx}f_y + 2f_x f_{xy}) + 2f_x^2 f_y)(ff_{xx} - f_x^2) \\
+ \quad & 3a(ff_{xy} - f_x f_y)(f^2 f_{xxx} - 3ff_x f_{xx} + 2f_x^3) \\
+ \quad & 2f^2(f^2 f_{xyt} - f(f_{xy}f_t + f_x f_{yt} + f_y f_{xt}) + 2f_x f_y f_t) \\
- \quad & 3f^2(f^2 f_{xxz} - f(f_{xx}f_z + 2f_x f_{xz}) + 2f_x^2 f_z) = 0. \qquad (2.109)
\end{aligned}$$

Since the left side of (2.109) is a polynomial in f, we set the coefficients to be 0 and get

$$a = 2, \qquad (2.110)$$

and

$$f_{xxxxy} + 2f_{xyt} - 3f_{xxz} = 0, \qquad (2.111a)$$

$$\begin{aligned}
-f_{xxxx}f_y - 4f_x f_{xxxy} + 2f_{xxx}f_{xy} - 2f_{xy}f_t & \\
-2f_x f_{xt} + 3f_{xx}f_z + 6f_x f_{xz} = 0, & \qquad (2.111b)
\end{aligned}$$

$$2f_x f_{xxx}f_y + 6f_x^2 f_{xxy} - 6f_x f_{xx}f_{xy} + 4f_x f_y f_t - 6f_x^2 f_z = 0. \qquad (2.111c)$$

Simplifying (2.111a)-(2.111c), we have

$$f_{xxxy} + 2f_{yt} - 3f_{xz} = 0, \qquad (2.112a)$$

$$f_{xxx}f_y - 3f_{xx}f_{xy} + 3f_x f_{xxy} + 2f_y f_t - 3f_x f_z = 0. \qquad (2.112b)$$

Let

$$f = \sum_{m=0}^{n} A_m(y,z,t)x^m, \qquad (2.113)$$

where

$$A_n = 1 \qquad (2.114)$$

and

$$A_m(y,z,t) = A_m(t) \qquad (2.115)$$

for $m = 1, \ldots, n-1$. We set $A_{n+s} = 0$ for $s > 0$. Then

$$f_x = \sum_{m=0}^{n} (m+1)A_{m+1}x^m, \qquad f_y = A_{0y}, \qquad (2.116)$$

$$f_t = \sum_{m=0}^{n} A_{mt}x^m, \qquad f_z = A_{0z}, \qquad (2.117)$$

$$f_{xx} = \sum_{m=0}^{n} (m+2)(m+1)A_{m+2}x^m, \qquad (2.118)$$

$$f_{xy} = f_{xz} = 0, \qquad f_{yt} = A_{0yt}, \tag{2.119}$$

and

$$f_{xxx} = \sum_{m=0}^{n} (m+3)(m+2)(m+1)A_{m+3}x^m. \tag{2.120}$$

Substituting (2.116)-(2.120) into (2.112a) and (2.112b), we get that

$$A_{0yt} = 0, \tag{2.121}$$

and

$$A_{0y}((m+3)(m+2)(m+1)A_{m+3} + 2A_{mt}) = 3(m+1)A_{0z}A_{m+1}. \tag{2.122}$$

Thus we can assume that

$$\frac{A_{0z}}{A_{0y}} = k, \tag{2.123}$$

where k is an constant.

By induction, we get that

$$A_{n-s} = \sum_{r=0}^{s} \sum_{p=0}^{[\![\frac{r}{2}]\!]} (\prod_{l=0}^{s-1}(n-l))(-1)^p \binom{s-r}{p} (\frac{1}{2})^p (\frac{3}{2}k)^{s-r-p} k_{n-r+2p} \frac{t^{s-r}}{(s-r)!} \tag{2.124}$$

for $s = 0, 1, \dots, n-1$. Here $k_n = 1$ and $k_1, \dots, k_{n-1}$ are arbitrary constants. Moreover,

$$\begin{aligned} A_0 &= \eta(y+kz) \\ &\quad + \sum_{r=0}^{n} \sum_{p=0}^{[\![\frac{r}{2}]\!]} (\prod_{l=0}^{n-1}(n-l))(-1)^p \binom{n-r}{p} (\frac{1}{2})^p (\frac{3}{2}k)^{n-r-p} k_{n-r+2p} \frac{t^{n-r}}{(n-r)!}, \end{aligned} \tag{2.125}$$

where $\eta(y+kz)$ is an arbitrary function of $y+kz$, and k_0 is an arbitrary constant.

In particular, we set

$$f = x + B(y, z, t). \tag{2.126}$$

By (2.112a) and (2.112b),

$$B_{yt} = 0, \tag{2.127a}$$
$$2B_yB_t = 3B_z. \tag{2.127b}$$

So, we have that

$$B = g(y, z) + h(t, z), \tag{2.128}$$

and

$$2g_yh_t = 3(g_z + h_z). \tag{2.129}$$

Assume that g is a polynomial in variable y. If

$$g = C(z)y + D(z), \tag{2.130}$$

then by (2.129),

$$2Ch_t = 3D_z + 3h_z, \tag{2.131}$$

and C is a constant. Differentiating (2.131) with respect to t, we obtain

$$\frac{2}{3}C = \frac{(h_t)_z}{(h_t)_t}. \tag{2.132}$$

Thus

$$h_t = \phi(t + \frac{2}{3}Cz), \tag{2.133a}$$

$$h_z = \frac{2}{3}c\phi(t + \frac{2}{3}Cz) + \psi'(z), \tag{2.133b}$$

where ϕ and ψ are arbitrary functions. Since

$$2C\phi(t + \frac{2}{3}Cz) - 2C\phi(t + \frac{2}{3}Cz) - 3\psi'(z) = 3D_z, \tag{2.134}$$

we have that

$$g = Cy - \psi(z) + k, \quad h = \rho(t + \frac{2}{3}Cz) + \psi(z). \tag{2.135}$$

where ρ and ψ are arbitrary functions.

If

$$g = \sum_{m=0}^{n} a_m(z)y^m, \qquad (n \geq 2) \tag{2.136}$$

then by (2.127b), we have

$$a_{n-m} = \sum_{r=0}^{m}(\prod_{s=0}^{m-1}(n-s))(\frac{2b}{3})^m k_r \frac{z^{m-r}}{(m-r)!} - \delta_{n,m}F(z) \tag{2.137}$$

for $m = 0, 1, \ldots, n-1$, and

$$h = bt + F(z), \tag{2.138}$$

where $F(z)$ is an arbitrary function.

Take

$$f = Ay + B. \tag{2.139}$$

According to (2.112a) and (2.112b),

$$A_x A_z = 0, \tag{2.140a}$$

$$A_{xxx} + 2A_t - 3B_{xz} = 0, \tag{2.140b}$$

$$AA_{xxx} + 2AA_t - 3(A_x B_z + A_z B_x) = 0, \tag{2.140c}$$

$$AB_{xxx} - 3A_x B_{xx} + 3A_{xx}B_x + 2AB_t - 3B_x B_z = 0. \tag{2.140d}$$

If $A_x = 0$, the solution will be the same as the preceding case. Thus we suppose

$$A_z = 0. \tag{2.141}$$

Moreover, we assume

$$A = e^{ax+bt}. \tag{2.142}$$

Then by (2.140b) and (2.140c), we obtain

$$B_{xz} = \frac{a^3+2b}{3}e^{ax+bt} \quad \text{and} \quad B_z = \frac{a^3+2b}{3a}e^{ax+bt}. \tag{2.143}$$

Thus

$$B = \frac{a^3+2b}{3a}Az + \phi(t,x). \tag{2.144}$$

Substituting (2.144) into (2.140d), we get

$$\phi_{xxx} - 3a\phi_{xx} + 3a^2\phi_x + 2\phi_t - \frac{a^3+2b}{a}\phi_x = 0. \tag{2.145}$$

It is a flag type equation [17]. We can get a basis of its polynomial solution space as follows

$$\begin{aligned}\phi(t,x) \quad = \quad & \sum_{r_1,r_2,r_3=0}^{\infty}(-1)^{r_1+r_3}\frac{3^{r_2}a^{r_2}(a^3-b)^{r_3}}{2^{r_1+r_2}}\frac{\prod_{s=0}^{3r_1+2r_2+r_3+1}(n-s)}{(r_1+r_2+r_3)!} \\ & \times x^{n-3r_1-2r_2-r_3}t^{r_1+r_2+r_3}. \end{aligned}\tag{2.146}$$

We write the results in this subsection as follows

Theorem 2.7. *The functions*

$$\begin{aligned} W_1 \quad = \quad & 2\Big(\sum_{s=1}^{n}\sum_{r=0}^{s}\sum_{p=0}^{[\![\frac{r}{2}]\!]}\Big(\prod_{l=0}^{s}(n-l)\Big)(-1)^p\binom{s-r}{p}\Big(\frac{1}{2}\Big)^p\Big(\frac{3}{2}k\Big)^{s-r-p}k_{n-r+2p} \\ & \times\frac{t^{s-r}}{(s-r)!}x^{n-s}\Big)\Big(\sum_{s=0}^{n}\sum_{r=0}^{s}\sum_{p=0}^{[\![\frac{r}{2}]\!]}\Big(\prod_{l=0}^{s-1}(n-l)\Big)(-1)^p\binom{s-r}{p}\Big(\frac{1}{2}\Big)^p\Big(\frac{3}{2}k\Big)^{s-r-p} \\ & \times k_{n-r+2p}\frac{t^{s-r}}{(s-r)!}x^{n-s} + \eta(y+kz)\Big)^{-1} \end{aligned}\tag{2.147}$$

$$W_2 = 2\Big(x + Cy + k + \rho\Big(t + \frac{2}{3}Cz\Big)\Big)^{-1}, \tag{2.148}$$

$$W_3 = 2\Big(x + \sum_{m=0}^{n}\sum_{r=0}^{m}\Big(\prod_{s=0}^{m-1}(n-s)\Big)\Big(\frac{2b}{3}\Big)^m k_r\frac{z^{m-r}}{(m-r)!}y^{n-m} + bt\Big)^{-1} \tag{2.149}$$

and

$$W_4 = 2\frac{ae^{ax+bt}y + \frac{a^3+2b}{3}e^{ax+bt}z + \phi_x(t,x)}{e^{ax+bt}y + \frac{a^3+2b}{3a}e^{ax+bt}z + \phi(t,x)} \tag{2.150}$$

are solutions of (1.1)*, where $\rho(t+\frac{2}{3}Cz)$ is an arbitrary function of $t+\frac{2}{3}Cz$, $\eta(y+kz)$ is an arbitrary function of $y+kz$, the numbers C, k, k_r, a and b are constants, and the function ϕ is given by* (2.146)*.*

3 Konopelchenko-Dubrovsky Equations

3.1 Stable-Range Approach

By (1.2b), we take the potential form

$$u = W_x, \quad v = W_y. \tag{3.1}$$

Then the Konopelchenko-Dubrovsky equations (1.2a) and (1.2b) are equivalent to

$$W_{xt} - W_{xxxx} - 6bW_xW_{xx} + \frac{3}{2}a^2W_x^2W_{xx} - 3W_{yy} + 3aW_{xx}W_y = 0. \tag{3.2}$$

Suppose

$$W = Ax^2 + Bx + C \tag{3.3}$$

for some functions A, B and C in t and y. Note that

$$W_x = 2Ax + B, \quad W_{xx} = 2A, \quad W_y = A_yx^2 + B_yx + C_y, \tag{3.4}$$

$$W_{yy} = A_{yy}x^2 + B_{yy}x + C_{yy}, \quad W_{xt} = 2A_tx + B_t. \tag{3.5}$$

Substituting (3.3)-(3.5), we find

$$\begin{aligned} & 2A_tx + B_t - 12Ab(2Ax + B) + 3a^2A(2Ax + B)^2 \\ - \; & 3(A_{yy}x^2 + B_{yy}x + C_{yy}) + 6aA(A_yx^2 + B_yx + C_y) = 0. \end{aligned} \tag{3.6}$$

Hence

$$4a^2A^3 - A_{yy} + 2aAA_y = 0, \tag{3.7a}$$

$$2A_t - 24A^2b + 12a^2A^2B - 3B_{yy} + 6aAB_y = 0, \tag{3.7b}$$

$$B_t - 12ABb + 3a^2AB^2 - 3C_{yy} + 6aAC_y = 0, \tag{3.7c}$$

Observe that

$$A = \frac{1}{ay + \psi(t)} \tag{3.8}$$

and

$$A = \frac{1}{-2ay + \psi(t)} \tag{3.9}$$

are two of solutions of (3.7a), where $\psi(t)$ is an arbitrary function. Substituting these two solutions into (3.7b), we get that

$$B = f_{-1}(t)(ay + \psi)^{-1} + f_0 + f_4(t)(ay + \psi)^4 \tag{3.10}$$

or

$$B = f_{-1}(t)(-2ay + \psi)^{-1} + f_0 + f_1(t)(-2ay + \psi), \tag{3.11}$$

where $f_0=(\psi_t+12b)/(6a^2)$. Thus we have

$$\begin{aligned}
C &= \frac{f_{-1}^2}{4}(ay+\psi)^{-1}-\frac{1}{3a^2}(-\frac{1}{3}f_{-1}f_4-4bf_{-1}+2a^2f_{-1}f_0)\log(ay+\psi)\\
&-\frac{1}{2a^2}(\frac{1}{3}f_{(-1)t}-4bf_0+a^2f_0^2)(ay+\psi)+\frac{\phi(z,t)}{3a}(ay+\psi)^3\\
&+\frac{f_{-1}f_4}{2}(ay+\psi)^4+\frac{1}{10a^2}(\frac{4}{3}f_4\psi_t-4bf_4+2a^2f_0f_4)(ay+\psi)^5\\
&+\frac{f_{4t}}{54a^2}(ay+\psi)^6+\frac{f_4^2}{54}(ay+\psi)^9+\varsigma(z,t)
\end{aligned}\tag{3.12}$$

or

$$\begin{aligned}
C &= \frac{1}{4}f_{-1}^2(-2ay+\psi)^{-1}-\frac{1}{2a}\phi(z,t)\log(-2ay+\psi)\\
&+\frac{1}{8a^2}(-\frac{1}{3}f_{-1}\psi_t-4bf_{-1}+2a^2f_0f_{-1})\log^2(-2ay+\psi)\\
&+\frac{1}{4a^2}(\frac{f_{(-1)t}}{3}-4bf_0+a^2(2f_{-1}f_1+f_0^2))(-2ay+\psi)\\
&+\frac{1}{16a^2}(\frac{1}{3}(f_{0t}+f_1\psi_t)-4bf_1+2a^2f_0f_1)(-2ay+\psi)^2\\
&+\frac{1}{36a^2}(\frac{1}{3}f_{1t}+a^2f_1^2)(-2ay+\psi)^3+\varsigma(z,t),
\end{aligned}\tag{3.13}$$

where $\phi(z,t)$ and $\varsigma(z,t)$ are arbitrary functions.

Theorem 3.1. *The functions*

$$\begin{aligned}
W_1 &= (ay+\psi(t))^{-1}x^2+(f_{-1}(t)(ay+\psi)^{-1}+f_0+f_4(t)(ay+\psi)^4)x\\
&- \frac{f_{-1}^2}{4}(ay+\psi)^{-1}-\frac{1}{3a^2}(-\frac{1}{3}f_{-1}f_4-4bf_{-1}+2a^2f_{-1}f_0)\log(ay+\psi)\\
&-\frac{1}{2a^2}(\frac{1}{3}f_{(-1)t}-4bf_0+a^2f_0^2)(ay+\psi)+\frac{\phi(z,t)}{3a}(ay+\psi)^3\\
&+\frac{f_{-1}f_4}{2}(ay+\psi)^4+\frac{1}{10a^2}(\frac{4}{3}f_4\psi_t-4bf_4+2a^2f_0f_4)(ay+\psi)^5\\
&+\frac{f_{4t}}{54a^2}(ay+\psi)^6+\frac{f_4^2}{54}(ay+\psi)^9+\varsigma(z,t)
\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}
W_2 &= (ay+\psi(t))^{-1}x^2+(f_{-1}(t)(-2ay+\psi)^{-1}+f_0+f_1(t)(-2ay+\psi))x\\
&\frac{1}{4}f_{-1}^2(-2ay+\psi)^{-1}-\frac{1}{2a}\phi(z,t)\log(-2ay+\psi)\\
&+\frac{1}{8a^2}(-\frac{1}{3}f_{-1}\psi_t-4bf_{-1}+2a^2f_0f_{-1})\log^2(-2ay+\psi)\\
&+\frac{1}{4a^2}(\frac{f_{(-1)t}}{3}-4bf_0+a^2(2f_{-1}f_1+f_0^2))(-2ay+\psi)\\
&+\frac{1}{16a^2}(\frac{1}{3}(f_{0t}+f_1\psi_t)-4bf_1+2a^2f_0f_1)(-2ay+\psi)^2\\
&+\frac{1}{36a^2}(\frac{1}{3}f_{1t}+a^2f_1^2)(-2ay+\psi)^3+\varsigma(z,t)
\end{aligned}\tag{3.15}$$

are solutions of (3.2)*, where* $f_0 = (\psi_t + 12b)/(6a^2)$*. The functions* $\psi(t)$*,* $f_{-1}(t)$*,* $f_1(t)$*,* $f_4(t)$*,* $\phi(z,t)$ *and* $\varsigma(z,t)$ *are arbitrary functions.*

3.2 Logarithmic Stable-Range Approach

Assume

$$W = m \log f \tag{3.16}$$

for some real number m and function f in t, x and y. Then

$$u = W_x = m\frac{f_x}{f}, \quad v = W_y = m\frac{f_y}{f}, \tag{3.17}$$

Note that

$$(\frac{f_x}{f})_t = \frac{f_{xt}f - f_x f_t}{f^2}, \quad (\frac{f_x}{f})_x = \frac{f_{xx}f - f_x^2}{f^2}, \tag{3.18}$$

$$(\frac{f_x}{f})_{xx} = \frac{f^2 f_{xxx} - 3ff_x f_{xx} + 2f_x^3}{f^3}, \quad (\frac{f_y}{y})_y = \frac{f_{yy}f - f_y^2}{f^2}, \tag{3.19}$$

and

$$(\frac{f_x}{f})_{xxx} = \frac{f^3 f_{xxxx} - f^2(4f_x f_{xxx} + 3f_{xx}^2) + 12ff_x^2 f_{xx} - 6f_x^4}{f^4}. \tag{3.20}$$

Substituting (3.16)-(3.20) into (3.2), we find

$$\begin{aligned}
& (ff_{xt} - f_x f_t)f^2 - (f_{xxxx}f^3 - (4f_x f_{xxx} + 3f_{xx}^2)f^2 + 12f_x^2 f_{xx} f - 6f_x^4) \\
- \quad & 6mbff_x(f_{xx}f - f_x^2) + \frac{3}{2}a^2 m^2 f_x^2 (f_{xx}f - f_x^2) - 3f^2(f_{yy}f - f_y^2) \\
+ \quad & 3amff_y(f_{xx}f - f_x^2) = 0.
\end{aligned} \tag{3.21}$$

We assume that the coefficients of the polynomial with respect to f in the left side of (3.21) are 0. Then we get

$$m = \pm\frac{2}{a}, \tag{3.22}$$

and

$$f_{xt} - f_{xxxx} - 3f_{yy} = 0, \tag{3.23a}$$

$$-af_x f_t + 4af_x f_{xxx} + 3af_{xx}^2 \mp 12bf_x f_{xx} + 3af_y^2 \pm 6af_y f_{xx} = 0, \tag{3.23b}$$

$$(-af_{xx} \pm 2bf_x \mp af_y)f_x^2 = 0. \tag{3.23c}$$

Simplifying the above system, we obtain

$$f_{xt} - f_{xxxx} - 3f_{yy} = 0, \tag{3.24a}$$

$$-af_t + 4af_{xxx} \mp 12bf_{xx} + \frac{12b^2}{a}f_x = 0, \tag{3.24b}$$

$$f_{xx} \pm f_y \mp \frac{2b}{a}f_x = 0. \tag{3.24c}$$

The equations (3.24b) and (3.24c) imply (3.24a). Note

$$f_{xxx} = \frac{4b^2}{a^2} f_x - \frac{2b}{a} f_y \mp f_{xy} \tag{3.25}$$

by (3.24c). Then

$$\begin{aligned} f_{xxxx} &= \pm \frac{2b}{a} f_{xxx} \mp f_{xxy} \\ &= \pm \frac{8b^3}{a^3} f_x \mp \frac{4b^2}{a^2} f_y - \frac{4b}{a} f_{xy} + f_{yy}. \end{aligned} \tag{3.26}$$

Moveover, by (3.24b), (3.25) and (3.26), we get

$$\begin{aligned} f_{xt} &= 4f_{xxxx} \mp \frac{12b}{a} f_{xxx} + \frac{12b^2}{a^2} f_{xx} \\ &= \pm \frac{8b^3}{a^3} f_x \mp \frac{4b^2}{a^2} f_y - \frac{4b}{a} f_{xy} + 4f_{yy} \\ &= f_{xxxx} + 3f_{yy}. \end{aligned} \tag{3.27}$$

Thus the system (3.24) can be written as

$$f_{xx} \pm f_y = \pm \frac{2b}{b} f_x, \tag{3.28a}$$

$$f_t = 4f_{xxx} \mp \frac{12b}{a} f_{xx} + \frac{12b^2}{a^2} f_x. \tag{3.28b}$$

Let

$$m = \frac{2}{a}. \tag{3.29}$$

Then the system (3.28) becomes

$$f_{xx} + f_y = \frac{2b}{b} f_x, \tag{3.30a}$$

$$f_t = 4f_{xxx} - \frac{12b}{a} f_{xx} + \frac{12b^2}{a^2} f_x. \tag{3.30b}$$

Note that the case $m = -2/a$ can be translated into the case $m = 2/a$ if we set $h(x, y, t) = f(-x, -y, -t)$. Thus it is sufficiently to calculate the case $m = 2/a$. We assume

$$f = \sum_{m=0}^{n} a_m(y,t)\xi^m, \qquad \xi = x + \frac{2b}{a} y + \frac{12b^2}{a^2} t. \tag{3.31}$$

Then

$$f_x = \sum_{m=0}^{n-1} (m+1) a_{m+1} \xi^m, \quad f_{xx} = \sum_{m=0}^{n-2} (m+2)(m+1) a_{m+2} \xi^m, \tag{3.32}$$

$$f_{xxx} = \sum_{m=0}^{n-3} (m+3)(m+2)(m+1) a_{m+3} \xi^m, \tag{3.33}$$

$$f_t=\sum_{m=0}^{n-1}(\frac{12b^2}{a^2}(m+1)a_{m+1}+a_{mt})\xi^m. \tag{3.34}$$

By (3.30a) and (3.30b), we find

$$a_{my}=-(m+2)(m+1)a_{m+2}, \tag{3.35a}$$

$$a_{mt}=4(m+3)(m+2)(m+1)a_{m+3}-\frac{12b}{a}(m+2)(m+1)a_{m+2}, \tag{3.35b}$$

where we have supposed that $a_{n+l}=0$ for $l>0$. Hence

$$a_n=b_n \qquad \text{and} \qquad a_{n-1}=b_{n-1} \tag{3.36}$$

are constants. Let

$$\begin{cases} \eta=y+\frac{12b}{a}t \\ t=t \end{cases} \tag{3.37}$$

Then we get that

$$a_{m\eta}=-(m+2)(m+1)a_{m+2}, \tag{3.38a}$$

$$a_{mt}=4(m+3)(m+2)(m+1)a_{m+3} \tag{3.38b}$$

by (3.35a) and (3.35b).

Denote

$$d(m,k)=[\![k/m]\!], \quad r(m,k)=k-[\![k/m]\!] \tag{3.39}$$

for $m,k\in\mathbb{Z}^+$. Then by induction, we obtain

$$\begin{aligned} a_{n-k}(y,t)=&\sum_{p=0}^{d(3,k)-1}\sum_{q=0}^{d(2,3p+r(3,k))}\frac{(-1)^{d(2,3p+r(3,k))-q}4^{d(3,k)-p}}{(d(3,k)-p)!(d(2,3p+r(3,k))-q)!} \\ \times&\ (\prod_{l=3p+r(3,k)}^{l}(n-l)) \\ \times&\ (\prod_{s=2q+r(2,3p+r(3,k))}^{3p+r(3,k)-1}(n-s))b_{n-2q-r(2,3p+r(3,k))}\eta^{d(2,3p+r(3,k))-p}t^{d(3,k)-p} \\ +&\sum_{q=0}^{d(2,k)}\frac{(-1)^{d(2,k)-q}}{(d(2,k)-q)!}(\prod_{l=2q-r(2,k)}^{k-1}(n-l))b_{n-2q-r(2,k)}\eta^{d(2,k)-q}, \end{aligned} \tag{3.40}$$

where b_{n-m} are constants.

Theorem 3.2. *For any positive integer n, the functions*

$$W=\pm\frac{2}{a}\log(\sum_{m=0}^{n}a_m(\pm y,\pm t)(\pm(x+\frac{2b}{a}y+\frac{12b^2}{a^2}t))^m) \tag{3.41}$$

are solutions of (3.2)*, where a_m are given by* (3.40).

Next we assume

$$f = \sum_{m=0}^{n} A_m y^m, \tag{3.42}$$

where A_m are functions in t and x. Then by (3.28a) and (3.28b),

$$A_{nxx} = \frac{2b}{a} A_{nx}, \tag{3.43a}$$

$$A_{mxx} = \frac{2b}{a} A_{mx} - (m+1) A_{m+1}, \tag{3.43b}$$

$$A_{mt} = \frac{4b^2}{a^2} A_{mx} + \frac{4b}{a}(m+1) A_{m+1} - 4(m+1) A_{(m+1)x}, \tag{3.43c}$$

for $m = 0, \ldots, n-1$.

Write

$$A_m = g_m(x,t) \exp(\frac{2b}{a} x + \frac{8b^3}{a^3} t) \tag{3.44}$$

for $m = 0, \ldots, n$. Then by (3.43a), (3.43b) and (3.43c),

$$g_{mxx} = -\frac{2b}{a} g_{mx} - (m+1) g_{m+1}, \tag{3.45a}$$

$$g_{mt} = \frac{12b^2}{a^2} g_{mx} + \frac{12b}{a} g_{mxx} + 4 g_{mxxx}. \tag{3.45b}$$

We assume

$$g_{n-m}(x,t) = \sum_{s=0}^{m} B_s^{n-m}(t) x^s \tag{3.46}$$

for $m = 0, \ldots, n$, where B_s^{n-m} are functions in t. Thus

$$B^0_{s,t} = \frac{12b^2}{a^2}(s+1) B^0_{s+1} + \frac{12b}{a}(s+2)(s+1) B^0_{s+2} + 4(s+3)(s+2)(s+1) B^0_{s+3}, \tag{3.47a}$$

$$B_s^{n-m+1} = -\frac{(s+2)(s+1)}{n-m+1} B_{s+2}^{n-m} - \frac{2b}{a} \frac{s+1}{n-m+1} B_{s+1}^{n-m} \tag{3.47b}$$

Firstly,

$$g_0 = \sum_{s=0}^{n} B_s^0 x^s. \tag{3.48}$$

Note that we can write

$$B^0_{n-m} = (\prod_{l=0}^{m-1} (n-l)) \sum_{p=0}^{m} c_{n-p} d_{m-p}, \tag{3.49}$$

where c_{n-p} are constants, and

$$d_0 = 1, \quad d_1 = \frac{12b^2}{a^2} t. \tag{3.50}$$

Observe that

$$d_m = \frac{12b^2}{a^2}\int d_{m-1}\mathrm{d}t + \frac{12b}{a}\int d_{m-2}\mathrm{d}t + 4\int d_{m-3}\mathrm{d}t. \tag{3.51}$$

Thus we can write

$$d_m = \sum_{s=0}^{[\![\frac{2m}{3}]\!]} e_{m,s}12^{m-s}(\frac{b}{a})^{2m-3s}\frac{t^{m-s}}{(m-s)!}. \tag{3.52}$$

Then

$$e_{0,0} = 1, \tag{3.53a}$$

$$e_{0,p} = e_{l,-p} = e_{-l,p} = 0 \quad \text{for} \;\; p > 0 \text{ and } l > 0. \tag{3.53b}$$

and

$$e_{m,k} = e_{m-1,k} + e_{m-2,k-1} + \frac{1}{3}e_{m-3,k-2}. \tag{3.54}$$

for $m > 0$ and $k > 0$ again by (3.47a). Hence

$$e_{m,k} = \sum_{s=0}^{k}(\frac{1}{3})^s\binom{k-s}{s}\binom{m-k}{k-s}. \tag{3.55}$$

Thus

$$d_m = \sum_{k=0}^{[\![\frac{2m}{3}]\!]}\sum_{s=0}^{k}12^{m-k}(\frac{1}{3})^s\binom{k-s}{s}\binom{m-k}{k-s}(\frac{b}{a})^{2m-3k}\frac{t^{m-k}}{(m-k)!}. \tag{3.56}$$

So we have

$$\begin{aligned} B^0_{n-m} &= (\prod_{l=0}^{m-1}(n-l))\sum_{p=0}^{m}\sum_{k=0}^{[\![\frac{2m-p}{3}]\!]}\sum_{s=0}^{k}c_{n-p}12^{m-p-k} \\ &\times (\frac{1}{3})^s\binom{k-s}{s}\binom{m-p-k}{k-s}(\frac{b}{a})^{2m-2p-3k}\frac{t^{m-p-k}}{(m-p-k)!}, \end{aligned} \tag{3.57}$$

$$\begin{aligned} g_0 &= \sum_{m=0}^{n}B^0_{n-m}x^{n-m} \\ &= \sum_{m=0}^{n}(\prod_{l=0}^{m-1}(n-l))\sum_{p=0}^{m}\sum_{k=0}^{[\![\frac{2m-p}{3}]\!]}\sum_{s=0}^{k}c_{n-p}12^{m-p-k} \\ &\times (\frac{1}{3})^s\binom{k-s}{s}\binom{m-p-k}{k-s}(\frac{b}{a})^{2m-2p-3k}\frac{t^{m-p-k}}{(m-p-k)!}x^{n-m}. \end{aligned} \tag{3.58}$$

Now we calculate

$$g_{n-q} = \sum_{r=0}^{q}B^{n-q}_r x^r \tag{3.59}$$

for $q = 0, 1, \ldots, n-1$,

$$B_r^{n-q} = -\frac{(r+2)(r+1)}{n-q} B_{r+2}^{n-q-1} - \frac{2b}{a} \frac{r+1}{n-q} B_{r+1}^{n-q-1} \tag{3.60}$$

for $q = 0, 1, \ldots, n-1$. Thus

$$B_r^m = \sum_{s=0}^{2^m} \Big(\prod_{l=0}^{m+s} (r+l) \Big) \frac{\binom{m}{s}}{m!} \Big(\frac{2b}{a}\Big)^s B_{r+m+s}^0 \tag{3.61}$$

where

$$B_{n+l}^0 = 0 \tag{3.62}$$

for $l > 0$.

Theorem 3.3. *The functions*

$$\begin{aligned} W \;\; = \;\; & \pm \frac{2}{a} \log\Big[\sum_{s=0}^{n} B_s^0(\pm t)(\pm x)^s \exp\Big(\pm\Big(\frac{2b}{a}x + \frac{8b^3}{a^3}t\Big)\Big) \\ & + \sum_{m=1}^{n} \sum_{r=0}^{n-m} \sum_{s=0}^{2^m} \Big(\prod_{l=0}^{m+s} (r+l) \Big) \frac{\binom{m}{s}}{m!} \Big(\frac{2b}{a}\Big)^s \\ & \times B_{r+m+s}^0(\pm t) \exp\Big(\pm\Big(\frac{2b}{a}x + \frac{8b^3}{a^3}t\Big)\Big)(\pm y)^m \Big] \end{aligned} \tag{3.63}$$

are solutions of (3.2)*, where* B_s^0 *are given by* (3.57) *and* (3.62).

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